General Relativity $\boxed{\text{Week 9}}$

Last time: We saw that the Einstein vacuum eqns $Ric_{\alpha\beta}=0$ in wave coordinates $(g^{\alpha\beta}\Gamma^{\mu}_{\alpha\beta}=0)$ become a quasilinear system of wave equations: $\Box_g g_{\mu\nu} = Q_{\mu\nu}(g,\partial g)$.

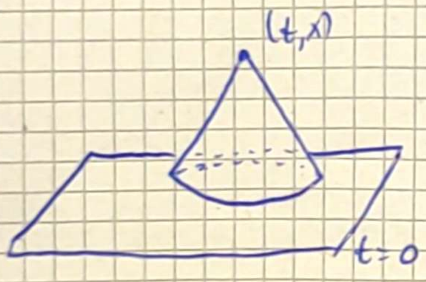
We want to understand the <u>initial value problem</u> for the Einstein eqns.

The simplest toy model:

$$\begin{cases} \Box_\eta \psi = 0 \quad \text{on } \mathbb{R}^{n+1} \\ \psi|_{t=0}=\psi_0, \quad \partial_t\psi|_{t=0}=\psi_1 \end{cases} \qquad \begin{aligned}\Box_\eta &= -\partial_t^2+\partial_1^2+\dots+\partial_n^2 \\ &= -\partial_t^2+\Delta_{\mathbb{R}^n}\end{aligned}$$

It can be solved explicitly:

eg when $n=3$, $$\psi(t,x) = \frac{1}{4\pi t^2}\int_{|y-x|=|t|}\left(t\cdot\psi_1(y)+\nabla\psi_0(y)\cdot(y-x)\right)dx$$



We can immediately deduce:

- Finite speed of propagation: if $\text{supp}\{\psi_0,\psi_1\}\subseteq B_R$, then $\text{supp}\{\psi|_{t=\tau}\}\subseteq B_{R+|\tau|}$
- Domain of dependence property:

  $\psi|_K$ depends only on the initial data on $J^-(K)\cap\{t=0\}$

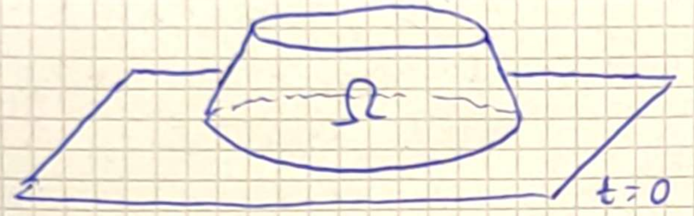
If we didn't know the explicit formula, how can we derive the same properties and obtain estimates for the solutions?

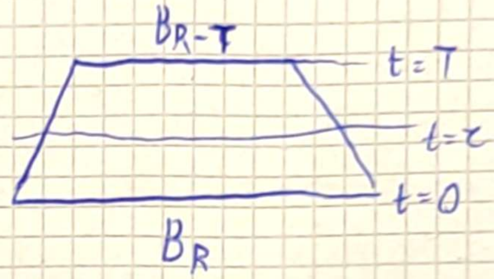
The fundamental energy identity:

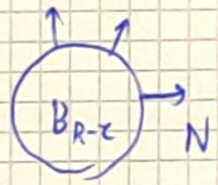
Assume that ψ is C^2, multiply with $\partial_t\psi$: $\longleftarrow$ ∂_t is timelike & Killing (algorithm related to Noether's thm).

$$0 = -\Box_\eta\psi\cdot\partial_t\psi = \frac{1}{2}\partial_t(\partial_t\psi)^2 - \mathrm{div}_x(\partial_t\psi\cdot\nabla\psi) + \frac{1}{2}\partial_t(|\nabla_x\psi|^2)$$

Integrate over spacetime truncated cone:







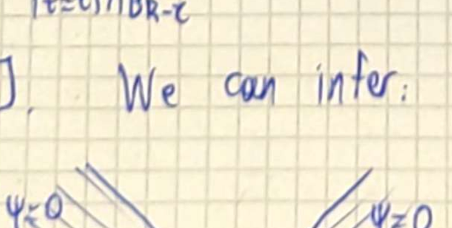
$$\Omega = \bigcup_{\tau\in[0,T]} \{\tau\}\times B_{R-\tau}, \quad T\leq R:$$

Then
$$0 = \int_\Omega -\Box_\eta\psi\cdot\partial_t\psi = \frac{1}{2}\int_{\{t=T\}\cap B_{R-T}} \left((\partial_t\psi)^2 + |\nabla_x\psi|^2\right)dx - \frac{1}{2}\int_{\{t=0\}\cap B_R}\left((\partial_t\psi)^2+|\nabla_x\psi|^2\right)dx$$
$$-\int_0^T \underbrace{\int_{\{t=\tau\}\cap S_{R-\tau}} \left[\frac{1}{2}\left((\partial_t\psi)^2 + |\nabla_x\psi|^2\right) - \partial_t\psi\cdot N\psi\right] d\sigma}_{\geq 0 \text{ by Cauchy-Schwarz}}\, d\tau$$

So if $\mathcal{E}[\tau] = \frac{1}{2}\int_{\{t=\tau\}\cap B_{R-\tau}}\left((\partial_t\psi)^2+|\nabla_x\psi|^2\right)dx$:

$\mathcal{E}[T]\leq\mathcal{E}[0]$. We can infer:

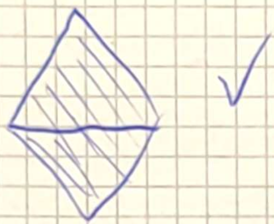
- Domain of dependence / finite speed
- Uniqueness of solutions:
  If $(\psi,\partial_t\psi)|_{t=0} = (0,0) \Rightarrow \psi\equiv 0$
- If compactly supp. initially $\Rightarrow$ compact supp. for all times



And crucially: Conservation of total energy for compactly supported initial data.

We also need to study the well-posedness of the initial value problem: Existence, uniqueness and continuous dependence of the solution on the initial data.

On what domains is the initial value problem well-posed?



Answer: Globally hyperbolic domains, with data on Cauchy hypersurface.

More generally: If (M^{n+1}, g) is Lorentzian, then the (inhomogeneous) covariant wave equation:

$$\Box_g \psi = F$$

where $\Box_g \psi := \mathrm{div}_g (d\psi) = g^{\mu\nu} (\underbrace{\nabla d\psi}_{\text{Hessian}})_{\mu\nu} = \nabla^a \nabla_a \psi$

$$= g^{\mu\nu} \partial_\mu \partial_\nu \psi - g^{\mu\nu} \Gamma^a_{\mu\nu} \partial_a \psi$$

$$= \frac{1}{\sqrt{-\det g}} \partial_\mu \left(g^{\mu\nu} \sqrt{-\det g}\, \partial_\nu \psi \right)$$

When $g = \eta$: In Cartesian coordinates this is the usual wave equation.

Initial value problem (Cauchy problem): $\begin{cases} \Box_g \psi = F & \text{on } M \\ \psi|_\Sigma = \psi_0, \ \hat{n}(\psi)|_\Sigma = \psi_1 \end{cases}$ ($\hat{n}$: timelike unit normal to Σ)

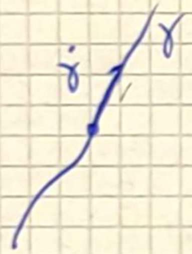
We will show: For $(\psi_0, \psi_1) \in C^\infty(\Sigma) \times C^\infty(\Sigma)$ and $F \in C^\infty(M)$: The above initial value problem is well-posed when Σ is a Cauchy hypersurface.

<u>Remark</u>: The cov. wave equation $\Box_g \psi = 0$ comes from a Lagrangian (i.e. it is the associated Euler-Lagrange equation):
Formally, solutions of $\Box_g \psi = 0$ are stationary points of the functional

$$\mathcal{L}[\psi] := \int_M g^{\mu\nu} \partial_\mu \psi \cdot \partial_\nu \psi \cdot \sqrt{-\det g}\, dx = \int_M |d\psi|_g^2 \, dvol_g$$

The corresponding field theory provides us with the associated <u>energy-momentum</u> tensor: $T[\psi] = d\psi \otimes d\psi - \frac{1}{2} \langle d\psi, d\psi \rangle_g \cdot g$

In local coordinates: $T_{\mu\nu}[\psi] = \partial_\mu \psi \, \partial_\nu \psi - \frac{1}{2} \partial^a \psi \cdot \partial_a \psi \cdot g_{\mu\nu}$



If γ is a timelike path (traced by an observer): $T_{\mu\nu} \dot\gamma^\mu \dot\gamma^\nu$ is the energy density of the scalar field as measured by γ.

We will use T independently of its physical interpretation (but we will borrow the language).

<u>Properties</u>: • If $\Box_g \psi = F$ then $\nabla^\mu T_{\mu\nu} = F \cdot \partial_\nu \psi$

So if $\Box_g \psi = 0 \Rightarrow \nabla^\mu T_{\mu\nu} = 0$ ("local conservation of energy/momentum")

(Proof: Use the symmetry of the Hessian $(\nabla d\psi)_{\alpha\beta} = (\nabla d\psi)_{\beta\alpha}$)

• If $X, Y \in T_pM$ are <u>future directed</u> and <u>timelike</u>:

$\forall \psi \in C^1(M)$ $\quad T_{\mu\nu}[\psi] X^\mu Y^\nu \geq c \cdot \sum_{a=0}^{n} (\partial_a \psi)^2$

↳ Depends only on X, Y, g, coordinates <u>not</u> on ψ.

Proof: • Easy if X, Y colinear (choose coordinates where $X = \partial_0$)

• Else, if the 2-plane ~~spanned~~ by $\{X, Y\}$

is spanned by the future directed null vectors $L, \underline{L}$

normalized so that $\langle L, \underline{L} \rangle = -2$, then

$$X = aL + b\underline{L} \qquad a,b,c,d > 0$$
$$Y = cL + d\underline{L}$$

Let $\{e_2, \dots, e_n\}$ be an orthonormal basis of $\{X, Y\}^\perp$ (that's a spacelike hyperplane), then in the frame $\{L, \underline{L}, e_2, \dots, e_n\}$:

$$T(L, L) = T_{\mu\nu} L^\mu L^\nu = (L\psi)^2$$
$$T(\underline{L}, \underline{L}) = (\underline{L}\psi)^2$$
$$T(L, \underline{L}) = \sum_{i=2}^{n} (e_i(\psi))^2$$

Def: Let X be a vector field on M. The current associated to X: $J^{(X)}_\mu[\psi] := T_{\mu\nu}[\psi] X^\nu$

1-form

Then $\boxed{\nabla^\mu J^{(X)}_\mu[\psi] = \Box_g \psi \cdot X(\psi) + T_{\mu\nu} \cdot \nabla^\mu X^\nu}$

By symmetry of $T_{\mu\nu}$: $= \frac{1}{2}(\nabla^\mu X^\nu + \nabla^\nu X^\mu)$
$= {}^{(X)}\pi^{\mu\nu} = \frac{1}{2}(\mathcal{L}_X g)^{\mu\nu}$

Deformation tensor

So: If $\Box_g \psi = 0$ and X is Killing:

$J^{(X)}_\mu[\psi]$ is conserved, i.e. divergence free (Noether's thm).

• In the case of $\Box_\eta \psi = 0$: We integrated the above identity for $X = \partial_t$ to obtain the energy identity!

We want to do something similar for the covariant wave equation on a spacetime without necessarily any Killing v.f.s

The divergence identity:

- Recall that, on $\mathbb{R}^n$, with the Euclidean (Riemannian) metric:

$$\int_\Omega \operatorname{div} X = \int_{\partial\Omega} \langle \hat{n}, X \rangle$$

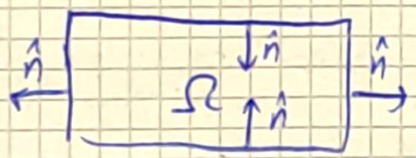
outward pointing unit normal

- On a general manifold of dimension m: If ω is an $m-1$ form

$$\int_\Omega d\omega = \int_{\partial\Omega} \omega \qquad \text{(Stokes theorem)}$$

exterior derivative

Proposition: Let $\Omega \subseteq (M, g)$ be a domain such that $\bar{\Omega}$ is compact, $\partial\Omega$ is piecewise smooth and has only spacelike and timelike components (i.e. not null). Choose $\hat{n}$ according to



Let ω be an 1-form on Ω (e.g. $J^{(X)}_\mu[\psi]$). Then

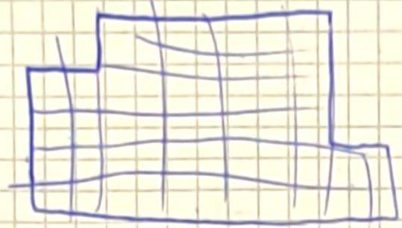
$$\int_\Omega \operatorname{div} \omega \, dvol_g = \int_{\partial\Omega} \omega(\hat{n}) \, dvol_{\bar{g}} \quad \leftarrow \bar{g}\text{: Induced metric on } \partial\Omega$$

Remark: One can also treat the case when $\partial\Omega$ has null pieces (e.g. the truncated cone in Minkowski). In that case, $\hat{n}$ and $dvol_{\partial\Omega}$ are not uniquely defined, but the "product" $\hat{n} \cdot dvol_{\partial\Omega}$ is (Given a choice of null normal $\hat{n}$: define $dvol_{\partial\Omega}$ so that

$$dvol_{\partial\Omega} = i_{\hat{n}} \, dvol_\Omega \quad)$$

interior product

Proof:



Ω

Assume without loss of generality (by subdivision into smaller sets) that Ω lies inside a coordinate chart $(x^0, \dots, x^n)$ and is a coordinate rectangle of the form

$$\Omega = \bigcap_{a=0}^{n} \{A^a < x^a < B^a\}$$

Note: $d\mathrm{vol}_g = \sqrt{-\det g}\, dx^0 \dots dx^n$

$$\mathrm{div}\, w = \nabla^\mu w_\mu = \frac{1}{\sqrt{-\det g}} \partial_\mu \left(\sqrt{-\det g}\, g^{\mu\nu} w_\nu\right)$$

So:

$$\int_\Omega \mathrm{div}\, w \, d\mathrm{vol}_g = \int_{A^n}^{B^n} \cdots \int_{A^0}^{B^0} \frac{1}{\cancel{\sqrt{-\det g}}} \partial_a \left(g^{a\beta} \sqrt{-\det g}\, w_\beta \right) \cancel{\sqrt{-\det g}}\, dx^0 \dots dx^n$$

$$= \sum_{a=0}^{n} \left(\int_{\{x^a = B^a\} \cap \partial\Omega} g^{a\beta} \sqrt{-\det g}\, w_\beta \, dx^0 \dots \widehat{dx^a} \dots dx^n - \int_{\{x^a = A^a\} \cap \partial\Omega} g^{a\beta} \sqrt{-\det g}\, w_\beta \, dx^0 \dots \widehat{dx^a} \dots dx^n \right)$$

The normal to $\{x^a = \text{const}\}$: has to be parallel to $(dx^a)^\#$

$$((dx^a)^\#)^\mu = g^{\mu\nu} \partial_\nu x^a = g^{\mu a}.$$

So: $\begin{cases} \hat{n}^\mu = \lambda \cdot g^{\mu a} \\ \hat{n}: \text{unit} \end{cases} \Rightarrow \hat{n}^\mu = \pm \frac{1}{\sqrt{|g^{aa}|}} g^{a\mu}$

$(g^{aa} > 0$: spacelike, $g^{aa} < 0$: timelike$)$

- If we denote with $\hat{g}_{(a,\beta)}$ the matrix of g without the a row and β column:

$$g^{a\beta} = (-1)^{a+\beta} \frac{\det(\hat{g}_{(a,\beta)})}{\det g}$$

So: $g^{aa} \cdot \det g = \det(\hat{g}_{(a,a)}) = \det \bar{g}_{\{x^a = const\}}$

Overall: $g^{a\beta} \omega_\beta \sqrt{-\det g} = \hat{n}^\beta \omega_\beta \sqrt{\pm \det \bar{g}}$ $\blacksquare$

<u>Remark:</u> Alternative proof using Stokes theorem:

The Hodge star operator $\star \; \Lambda^k M \longrightarrow \Lambda^{\dim M - k} M$

Satisfies:

- $\star\star = (-1)^{k\cdot(m-k)+1}$
- $\langle \star \omega_1, \omega_2 \rangle = (-1)^{k\cdot(m-k)+1} \langle \omega_1, \star \omega_2 \rangle$
- $\star 1 = dvol_g$
- $div_g = - \star^{-1} d \star$

Then: $\int_\Omega div_g \, \omega \cdot dvol_g = \langle - \star^{-1} d \star \omega, \star 1 \rangle$

$$= \pm \int_\Omega d \star \omega = \pm \int_{\partial\Omega} \star \omega \quad \ldots$$